5 SEM TDC MTMH (CBCS) C 12

2024

(November)

MATHEMATICS

(Core)

Paper : C-12

(Group Theory—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. (a) State True or False : 1

Every isomorphism is an automorphism.

(b) Show that a normal subgroup of a group G may not be a characteristic subgroup of G. 2

(c) Let $f: G \to G$ be a homomorphism. Suppose f commutes with every inner automorphism of G. Show that

$$K = \{x \in G \mid f^2(x) = f(x)\}$$

is a normal subgroup of G. 3

(d) If $f : G \to G$ such that $f(x) = x^n$ is an automorphism, where n is some fixed integer, then show that

$$a^{n-1} \in Z(G)$$

for all $a \in G$. 3

(e) Let G be a group. Show that the mapping $\phi : G \to G$ such that

$$\phi(x) = x^{-1} \ \forall x \in G$$

is an automorphism if and only if G is abelian. 4

(f) Show that the set $I(G)$ of all inner automorphisms of G is a subgroup of aut G. 5

2. Answer any *two* of the following : 6×2=12

(a) For every positive integer n, prove that aut (Z_n) is isomorphic to $U(n)$.

(b) Let G' be the commutator subgroup of a group G. Then show that—

(i) G' is normal in G;

(ii) $\dfrac{G}{G'}$ is abelian.

(c) If N is a normal subgroup of a group G and G' be the commutator subgroup of G and $N \cap G' = \{e\}$, then show that

$$N \subseteq Z(G)$$

3. (a) Express $U(105)$ as an external direct product of U groups in three different ways. 3

(b) Find the number of elements of order 5 in $Z_{25} \oplus Z_5$. 3

(c) Let $G = \langle a \rangle$ be an abelian group of order 6. Let

$$H = \{e, a^2, a^4\}, K = \{e, a^3\}$$

Then prove that H and K are normal subgroups of G. 4

(d) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 5

Or

Let G be a group and suppose G is IDP of $H_1, H_2, \cdots, H_n$. Let T be EDP of $H_1, H_2, \cdots, H_n$. Then show that G and T are isomorphic.

(e) Show that every group of order p^2, where p is a prime, is either cyclic or isomorphic to direct product of two cyclic groups, each of order p. 5

Or

Prove that a group G is internal direct product of its subgroups H and K if and only if (i) H and K are normal subgroups of G and (ii) $H \cap K = \{e\}$.

4. (a) Define Sylow p-subgroup. 1

(b) What is a simple group? Give one example of a simple group. 1+1=2

(c) If a be an element of a group G, then show that $Cl(a) = \{a\}$ if and only if $a \in Z(G)$. 3

(d) Prove that a group of order p^2 is abelian. 4

(e) Let G be a finite group whose order is a prime p. Then show that $Z(G)$ has more than one element. 4

Or

Let G be a finite group and p a prime that divides the order of G. Then show that G has an element of order p.

(f) Let G be a finite group and $a \in G$. Then prove that

$$O(Cl(a)) = \frac{O(G)}{O(N(a))}$$

where $Cl(a)$ is the conjugate class of a. 5

(g) Prove that a group of order 15 is abelian. 5

Or

Prove that no group of order 30 is simple.

(h) If H is a subgroup of a finite group G and $|H|$ is a power of a prime p, then prove that H is contained in some p-subgroup of G. 6

Or

Show that the number of Sylow p-subgroups of a group G is equal to 1 modulo p and divides $|G|$. Also show that any two Sylow p-subgroups of G are conjugate.

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