Total No. of Printed Pages—7

5 SEM TDC MTH M 1

2016

(November)

MATHEMATICS

(Major)

Course : 501

(Logic and Combinatorics, and Analysis—III)

Full Marks : 80

Pass Marks : 32 (Backlog)/24 (2014 onwards)

Time : 3 hours

The figures in the margin indicate full marks for the questions

(A) Logic and Combinatorics

(*Marks* : 35)

1. (*a*) State 'True' or 'False' : 1×2=2

(*i*) '$x-4=6$' is a statement.

(*ii*) 'What is your name?' is a statement.

(*b*) (*i*) Write down the converse of $(p \to q)$. 1

(*ii*) Find the dual of $\sim(p \wedge q) \vee T$. 2

(c) (i) Prove that $p \to q \equiv \sim p \vee q$. 1

(ii) Prove that the set $\{\to, \sim\}$ is functionally complete. 4

Or

Using arithmetical representation, prove that $A \vee (A \leftrightarrow A)$ is a tautology. 4

2. (a) Define rules of inferences. 2

(b) Illustrate the derivation

$$A \to B, \sim (B \vee C) \models \sim A$$ 2

(c) Symbolize the following sentence using predicates : 2

"There are both lawyers and shysters who admire John."

(d) If P_x be 'x is prime', O_x be 'x is odd', D_{xy} be 'x divides y', then translate the following into English : 4

$$(x)(O_x \to (y)(P_y \to \sim D_{xy}))$$

Or

Write the formal derivation of the following sentence : 4

"No human beings are quadrupeds.
All women are human beings.
Therefore, no woman is quadruped."

3. (a) State the rules of sum and product of counting. 1

(b) In how many ways can we get a total of six while rolling two dice simultaneously? 2

Or

How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are non-negative integers? 2

(c) State Vandermonde's identity. Prove that

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

where n, r are non-negative integers such that $r \leq n$. 1+3=4

4. (a) Define Ramsey number. Show that

$$R(m, n) \leq C(m+n-2, m-1)$$

where m, n are integers greater than 1.

1+3=4

Or

Show that

(i) $R(4, 4) = 18$

(ii) $R(5, 3) = 14$ 2+2=4

(b) How many integers between 1 and 500 are *(i)* divisible by 3 or 5 and *(ii)* divisible by 3 but not by 5 or 6? 4

Or

Find a generating function for a_r = the number of non-negative integral solutions to $e_1 + e_2 + \cdots + e_n = r$, where $0 \le e_i$ for each i. 4

(B) Analysis—III (Complex Analysis)

(*Marks* : 45)

5. *(a)* Write down the conditions for any complex function to be analytic. 1

(b) Derive Cauchy-Riemann equation for a complex function $f(z)$ in Cartesian coordinates. 3

(c) Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}};\ z \neq 0,\ f(0) = 0$$

in a region including the origin. 6

Or

Show that the function $f(z) = z^3$ is analytic in a domain D of a complex plane C. 6

6. *(a)* Define rectifiable curve. 1

(b) Show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \overline{z}} \qquad 4$$

(c) If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of z, find $f(z)$ in terms of z. 5

Or

State and prove Cauchy's theorem. 5

(d) Answer the following (any *one*) : 4

(i) Evaluate

$$\int_C \frac{dz}{z(z-1)}$$

where C is the circle $|z| = 3$.

(ii) Evaluate

$$\int_C \frac{z-1}{(z+1)^2 (z-2)} dz$$

where C is such that $|z - i| = 2$.

7. (a) Define singularities of an analytic function. 2

(b) Expand

$$\frac{1}{z(z^2 - 3z + 2)}$$

for the region $0 < |z| < 1$. 3

(c) Expand e^z in a Taylor's series about $z=0$ and determine the region of convergence. 3

Or

Find Taylor's expansion of $f(z)=\dfrac{z}{z^4+9}$ about $z=0$. 3

8. (a) Find the residues of the function

$$f(z)=\frac{\cot \pi z}{(z-a)^2}$$ 3

(b) Evaluate the following (any *two*) : 5×2=10

(i) $\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta+\sin\theta)\,d\theta$

where n is a positive integer

(ii) $\int_0^{2\pi}\dfrac{d\theta}{1+a^2-2a\cos\theta}$

(iii) $\int_{-\infty}^{\infty}\dfrac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$; $a>b>0$

(iv) $\int_{-\infty}^{\infty}\dfrac{x^2}{(x^2+a^2)^3}dx$

where residue is taken to be positive

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5 SEM TDC MTH M 2

2016

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32 (Backlog)/24 (2014 onwards)

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. (a) Write which of the following statements is 'true' and which is 'false' : 1×2=2

(i) "The set containing a linearly independent set of vectors is itself linearly independent."

(ii) "Intersection of two subspaces of a vector space V is always a subspace of V."

(b) Examine whether the vector (2, −5, 3) is in the subspace of $\mathbb{R}^3$ spanned by the vectors (1, −3, 2), (2, −4, −1) and (1, −5, 7). 3

(c) Show that the set

$$S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$$

forms a basis for the vector space V of ordered pairs of complex numbers over the field of real numbers $\mathbb{R}$, i.e., $V = \mathbb{C}^2(\mathbb{R})$. 3

(d) Let V be a finite dimensional vector space of dimension n. Then prove that any set of n linearly independent vectors in V forms a basis for V. 3

(e) Let V be any vector space. Prove that the set $\{v_1, v_2, ..., v_n\}$ is linearly dependent if and only if one of the v_i's is a linear combination of the other v_j's where $v_k \in V$, $1 \leq k \leq n$. 5

(f) Define subspace of a vector space. Prove that the set W defined as

$$W = \{(a, b, 0) : a, b \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3$. 2+2=4

(b) Examine whether the vector $(2, -5, 3)$ is in the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7)$. 3

(c) Show that the set

$$S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$$

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(d) Let V be a finite dimensional vector space of dimension n. Then prove that any set of n linearly independent vectors in V forms a basis for V. 3

(e) Let V be any vector space. Prove that the set $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if and only if one of the v_i's is a linear combination of the other v_j's where $v_k \in V$, $1 \le k \le n$. 5

(f) Define subspace of a vector space. Prove that the set W defined as

$$W = \{(a, b, 0) : a, b \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3$. 2+2=4

2. (a) Let $l(p; d)$ and $l(q; d)$ be two lines passing through p and q respectively having direction d. Show that $l(p; d) = l(q; d)$ if and only if $(q-p)$ is a multiple of d. 3

(b) Let T be a linear transformation from a vector space U to a vector space V over the field F. Prove that the range of T is a subspace of V. 3

(c) Show that a linear map T from a vector space to another is one-one if and only if $\ker T = \{0\}$. 4

(d) Let V be the vector space of all polynomials in x with coefficients in $\mathbb{R}$ of the form

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

The differentiation operator D is a linear transformation on V. Write the matrix of D relative to the ordered basis

$$B = \{x^0, x^1, x^2, x^3\}$$ 4

(e) Show that the mapping $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(a, b) = (a+b, a-b, b)$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Find the rank and nullity of T. 2+2+2=6

Group—B

(Number Theory)

(*Marks* : 40)

3. When are two integers said to be relatively prime? 1

4. Answer any *two* from the following : 3×2=6

(*a*) Use division algorithm to establish that the square of any integer is either of the form $3k$ or $3k+1$.

(*b*) Prove that if $a \mid bc$ with $\gcd(a, b) = 1$, then $a \mid c$.

(*c*) Use Euclidean algorithm to obtain integers x and y satisfying the following :

$$\gcd(56, 72) = 56x + 72y$$

5. (*a*) Show that if p is a prime and $p|ab$ then either $p|a$ or $p|b$. 3

(*b*) Prove that given any positive integer n, there exist n consecutive composite integers. 3

(*c*) Find the highest power of 5 dividing $100!$. 2

6. (a) Write a complete set of residues modulo 7. 1

(b) If $a \equiv b \pmod{n}$ and the integers a, b, n are all divisible by $d > 0$, then prove that

$$\frac{a}{d} \equiv \frac{b}{d}\left(\bmod \frac{n}{d}\right)$$ 3

(c) If a is an odd integer, then prove that

$$a^2 \equiv 1 \pmod{8}$$ 4

(d) Solve $18x + 5y = 48$. 4

(e) Solve the following by using Chinese remainder theorem : 3

$$x \equiv 5 \pmod{4}$$
$$x \equiv 3 \pmod{7}$$
$$x \equiv 2 \pmod{9}$$

7. (a) Evaluate

(i) $\sigma(210)$

(ii) $d(63)$

(iii) $\phi(100)$

where the symbols have their usual meanings. 2×3=6

(b) When is an arithmetic function said to be multiplicative? Prove that σ is a multiplicative function. 1+3=4

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5 SEM TDC MTH M 3

2016

(November)

MATHEMATICS

(Major)

Course : 503

(Fluid Mechanics)

Full Marks : 80

Pass Marks : 32 (Backlog) / 24 (2014 onwards)

Time : 3 hours

The figures in the margin indicate full marks for the questions

(A) Hydrodynamics

(*Marks* : 35)

1. (a) Define ideal fluid. 1

(b) State whether True or False : 1

A path line is the curve along which a particular fluid particle travels during its motion.

(c) Find the equation of the streamlines for the flow $\vec{q} = -\hat{i}(3y^2) - \hat{j}(6x)$ at the point (1, 1). 3

(d) Determine the acceleration at the point (2, 1, 3) at $t = 0\cdot5$ if $u = yz + t$, $v = xz - t$ and $w = xy$. 4

2. Deduce the equation of continuity in cylindrical coordinates. 6

Or

Show that

$$u = \frac{-2xyz}{(x^2+y^2)^2},\ v = \frac{(x^2-y^2)z}{(x^2+y^2)^2} \text{ and } w = \frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion. Is this motion irrotational? 6

3. *(a)* Choose the correct answer : 1

Euler's equation of motion in x direction is

(i) $\dfrac{Du}{Dt} = X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

(ii) $\dfrac{Du}{Dt} = X + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

(iii) $\dfrac{\partial u}{\partial t} = X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

(iv) $\dfrac{\partial u}{\partial t} = X + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

(b) If the motion of an ideal fluid, for which density is a function of pressure only, is steady and the external forces are conservative, then prove that there exists a family of surfaces which contain the streamlines and vortex lines. 5

Or

For a steady motion of inviscid incompressible fluid under conservative forces, show that the vorticity $\vec{\omega}$ and velocity $\vec{q}$ satisfies

$$(\vec{q} \cdot \nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{q}$$ 5

4. State and prove Kelvin's circulation theorem. 6

Or

A portion of homogeneous fluid is contained between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance. The inner spherical surface is suddenly annihilated, and when the radii of inner and outer surface of the fluid are r and R, the fluid impinges on a solid ball concentric with these surfaces. Prove that the impulsive pressure at any point of the ball for different values of R and r varies as

$$\left\{(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r} - \frac{1}{R}\right)\right\}^{\frac{1}{2}}$$ 6

5. (a) Define circulation. 1

(b) Answer *either* (i) *or* [(ii) and (iii)]

(i) Show that if the velocity potential of an irrotational motion is equal to

$$A(x^2+y^2+z^2)^{-\frac{3}{2}}\left(z\tan^{-1}\frac{y}{x}\right)$$

the lines of flow lie on the family of surfaces

$$x^2+y^2+z^2=k^{\frac{2}{3}}(x^2+y^2)^{\frac{2}{3}}$$ 7

Or

(ii) Prove that there cannot be two different forms of irrotational motion for a given confined mass of incompressible inviscid liquid whose boundaries are subject to the given impulses. 3

(iii) If Σ is the solid boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surfaces, then show that the mean value of velocity potential Q on Σ is of the form

$$Q=\left(\frac{M}{R}\right)+C$$

where M, C are constants, provided that the fluid extends to infinity and is at rest there. 4

(B) Hydrostatics

(*Marks* : 45)

6. (a) Define specific gravity of a substance. 1

(b) Prove that the densities at two points in a fluid at rest under gravity and in the same horizontal plane are equal. 2

(c) Prove that the surfaces of equal pressure are intersected orthogonally by the lines of force. 3

7. (a) A tube in the form of a parabola held with its vertex downwards and axis vertical, is filled with different liquids of densities δ and δ'. If the distance of the free surface of the liquids from the focus be r and r' respectively, show that the distance of their common surface from the focus is

$$\frac{r\delta - r'\delta'}{\delta - \delta'}$$ 6

Or

If the components parallel to the axes of the forces acting on an element of fluid at (x, y, z) be proportional to $y^2 + 2\lambda yz + z^2$, $z^2 + 2\mu zx + x^2$ and

$$x^2 + 2\nu xy + y^2$$

show that if equilibrium be possible, then $2\lambda = 2\mu = 2\nu = 1$. 6

(*b*) Prove that the pressure at a depth z below the surface of a homogeneous liquid, at rest under gravity is $p = wz + \Pi$, where Π is the atmospheric pressure and w is the weight of unit volume of the liquid. 5

8. (*a*) Define centre of pressure. 1

(*b*) Prove that the whole pressure of a heavy homogeneous liquid on a plane area is equal to the product of the area and the pressure at its centre of gravity. 3

9. (*a*) Find the centre of pressure of a parallelogram immersed in a homogeneous liquid with one side in the free surface. 6

Or

A triangle ABC is immersed in a liquid, its plane being vertical and the side AB in the surface; if O be the centre of the circumscribed circle of ABC, prove that

$$\frac{\text{Pressure on the } \Delta AOC}{\text{Pressure on the } \Delta OCB} = \frac{\sin 2B}{\sin 2A}$$ 6

(*b*) A conical glass is filled with water and placed in an inverted position upon a table. Show that the resultant vertical thrust of the water on the glass is two-thirds that on the table. 6

Or

Find the resultant horizontal thrust in an assigned horizontal direction on a curved surface immersed in a heavy homogeneous liquid. 6

10. *(a)* Fill in the blank : 1

If the ____ coincides with centre of gravity, the equilibrium is neutral.

(b) A body floats partly immersed in one liquid and partly in another. Find the condition of equilibrium. 4

(c) Define stable and unstable equilibrium. 2

11. Prove that the tangent at any point of surface of buoyancy is parallel to the corresponding plane of floatation. 5

Or

A solid body consists of a right cone joined to hemisphere on the same base and floats with the spherical portion partly immersed. Prove that the greatest height of the cone consistent with stability is $\sqrt{3}$ times the radius of the base. 5

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5 SEM TDC MTH M 4

2016

(November)

MATHEMATICS

(Major)

Course : 504

(Mechanics and Integral Transform)

Full Marks : 80

Pass Marks : 32 (Backlog)/24 (2014 onwards)

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Mechanics)

(a) : Statics

(*Marks* : 25)

1. *(a)* Write to which a system of forces acting at different points of a rigid body can be reduced. 1

(b) Define wrench. 2

(c) Find the necessary and sufficient conditions for equilibrium of a rigid body. 7

Or

Find the equation of the central axis of a system of forces acting on a rigid body.

2. (a) Define axis of catenary. 1

(b) Write the height of the centre of gravity of a body for stable equilibrium. 1

(c) Establish the relation between x and s in a common catenary. 2

(d) Show that the total virtual work done by tensions of an inextensible string is zero. 5

Or

Five weightless rods of equal length are joined together to form a rhombus *ABCD* with one diagonal *BD*. If a weight w is attached to *C* and the system be suspended from *A*, show that there is a thrust in *BD* equal to $\frac{w}{\sqrt{3}}$.

(e) State and prove the principle of virtual work for a system of coplanar forces acting at different points of a rigid body. 6

Or

Derive the equation $s = c\tan\psi$ for common catenary.

(b) : Dynamics

(*Marks* : 25)

3. (*a*) Let $v = ft$. Show that the acceleration f is constant. 2

(*b*) Find the components of velocity along and perpendicular to the radius vector of a particle moving in a plane curve. 6

Or

Let a particle move in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point. Find the equation of the path of the particle.

4. (*a*) Write to which velocity of a particle at any point in a central force varies. 1

(*b*) A particle moves in a plane with an acceleration which is always directed to a fixed point in the plane. Discuss the motion. 6

Or

A particle is falling under gravity in a medium whose resistance varies as the velocity. Find the distance at any instant of time.

5. *(a)* Write the moment of inertia of mass m whose coordinates are (x, y, z) with respect to x-axis. 1

(b) Define momental ellipsoid of a body. 2

(c) Write the effective forces on a particle along the tangent and normal. 2

(d) Find the moment of inertia of a rectangular lamina about a line through its centre and parallel to one of its edges. 5

Or

Prove the theorem of perpendicular axes of moment of inertia.

GROUP—B

(Integral Transform)

(*Marks* : 30)

6. *(a)* Write the value of the following : 1+1+1=3

(i) $L\{t^2\}$

(ii) $L\{\cos 2t\}$

(iii) $L\{\sin^2 t\}$

(b) Find $L\{\cosh 2x\}$. 2

(c) Find $L\{t^2 \cos at\}$. 3

Or

Find $L\{\sinh at \sin at\}$.

7. *(a)* Write the value of $L^{-1}\left\{\frac{1}{s^2+9}\right\}$. 1

(b) Find : 2+2=4

(i) $L^{-1}\left\{\frac{1}{(s-4)^2}\right\}$

(ii) $L^{-1}\left\{\frac{s+4}{(s+4)^2+4}\right\}$

(c) Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$. 3

Or

Find $L^{-1}\left\{\frac{s-2}{s^2-2s+5}\right\}$.

8. (a) If $y = y(x, t)$, then write the value of $L\left\{\frac{\partial y}{\partial x}\right\}$. 1

(b) Solve any *two* of the following using Laplace transform : 4×2=8

(i) $(D^2+25)y = 10\cos 5t$,

$y(0)=2,\ y'(0)=0,\ \left(D \equiv \frac{d}{dt}\right)$

(ii) $(D^2+9)y = 6\cos 3t,\ y(0)=2,\ y'(0)=0$

(iii) $(D^2+2D+5)y = e^{-t}\sin t$,

$y(0)=3,\ y'(0)=1$

(c) Solve $\frac{\partial y}{\partial t}=\frac{\partial^2 y}{\partial x^2}$, if $y(x, 0)=3\sin 2\pi x$,

$y(0, t)=0, \quad y(1, t)=0$ 5

Or

Solve :

$$(D-2)x-(D+1)y=6e^{3t}$$
$$(2D-3)x+(D-3)y=6e^{3t}$$
$$x(0)=3, \quad y(0)=0$$

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