5 SEM TDC MTMH (CBCS) C 12

2022

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–12

(Group Theory—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Choose the correct answer for the following question : 1

An automorphism is

(i) a homomorphism but not one-one

(ii) a homomorphism, one-one but not onto

(iii) one-one, onto but not homomorphism

(iv) a homomorphism, one-one and onto

(b) Show that a characteristic subgroup of a group G is a normal subgroup of G. Is the converse true? 2+2=4

(c) Let G' be the commutator subgroup of a group G, then prove that G is abelian if and only if $G' = \{e\}$. 3

(d) If N is a normal subgroup of a group G, G' is the commutator subgroup of G and $N \cap G' = \{e\}$, then show that $N \subseteq Z(G)$. 3

(e) Show that, if $O(\text{Aut}\, G) > 1$ then $O(G) > 2$. 3

(f) Show that the set $I(G)$ of all inner automorphism of a group G is a subgroup of Aut G. 4

2. Answer any *two* of the following : 6×2=12

(a) Let $I(G)$ be the set of all inner automorphisms on a group G, then prove that

$$I(G) \approx \frac{G}{Z(G)}$$

(b) Prove that for every positive integer n, Aut (Z_n) is isomorphic to $U(n)$.

(c) Let $R^n = \{(a_1, a_2, ..., a_n) \mid a_i \in R\}$. Show that the mapping

$$\phi : (a_1, a_2, ..., a_n) \to (-a_1, -a_2, ..., -a_n)$$

is an automorphism of the group R^n under component wise addition.

3. (a) Find the order of the element (1, 1) in $Z_{100} \oplus Z_{25}$. 2

(b) Show that a group of order 4 is either cyclic or is an internal direct product of two cyclic groups of order 2 each. 3

(c) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 4

(d) If s and t are relatively prime, then prove that

$$U(st) \approx U(s) \oplus U(t)$$ 5

Or

How many elements of order 5 does $Z_{25} \oplus Z_5$ have?

(e) If a group G is the internal direct product of a finite number of subgroups $H_1, H_2, ..., H_n$, then prove that G is isomorphic to the external direct product of $H_1, H_2, ..., H_n$. 6

Or

Let G be a finite abelian group of order $p^n m$, where p is a prime that does not divide m then prove that $G = H \times K$, where $H = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$.

4. (a) Define conjugate class of a. 1

(b) If $|G| = p^2$, where p is a prime, then prove that G is abelian. 3

(c) Let G be a finite group and let a be an element of G, then prove that

$$|Cl(a)| = |G : C(a)|$$ 3

(d) Prove that a group of order 80 has a non-trivial normal Sylow p-subgroup. 3

(e) Let G be a group. Prove that $Cl(a) = \{a\}$, if and only if $a \in Z(G)$. 4

(f) Prove that no group of order 56 is simple. 5

Or

Prove that a Sylow p-subgroup of a group G is normal if and only if it is the only Sylow p-subgroup of G.

(g) If G is a group of order pq, where p and q are primes, $p < q$, and p does not divide $q - 1$, then prove that G is cyclic. 5

(h) Prove that any two Sylow p-subgroups of a finite group G are conjugate in G. 6

Or

Prove that an integer of the form $2 \cdot n$, where n is an odd number greater than 1, is not the order of a simple group.

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